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Quantum Geometric Description of Cosmological Models

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Abstract

This is a written version of the review talk given at the meeting on “Interface of Gravitational and Quantum Realms” at IUCAA, Pune during December 2001. The talk reviewed the recent work of Martin Bojowald on Loop Quantum Cosmology.

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I. INTRODUCTION

Canonical quantization of gravity formulated in terms of connection variable is now at a stage where one can address questions of physical interest. At the interpretational level, one still has the outstanding issues of understanding the classical limit and development of a systematic semiclassical expansion. [1]

There are at least two classes of situations in classical GR that call for a quantum elucidation: (i) situations where classical GR predicts ‘singularities’ eg early universe and collapse, and (ii) situations where GR predicts horizons possessing entropy. In a sense, these can be thought of as ‘requiring’ a quantum theory of gravity and hence any such theory should lead to better understanding of these situations.

There is another context in which one can look for signatures of quantum nature of space-time. This refers to matter wave propagation in a quantum geometrical background. Looking for such effects experimentally actually seems possible thanks to the GRB sources at cosmological distances. This context is very different from the first two in that QG effects are being probed in very ordinary, highly classical, non-extreme situations. [2]

The horizon context has already been analyzed and quantum geometry framework does provide a microscopic understanding of entropy. Hawking effect is however is not yet clear. [3]

Of the two singular context, Martin Bojowald has recently analyzed the cosmological context in detail. As this is a first application of the quantum geometry framework with very interesting results, I decided to review this work in some details.

II. STANDARD QUANTUM COSMOLOGY

A version of quantum gravity in the cosmological context was already attempted in the sixties. This was again a canonical quantization approach but using the geometrodynamical variables - metric g_{ij} and a symmetric second rank tensor k_{ij} defined on a three manifold. This is a constrained system with diffeomorphism constraint implementing spatial diffeomorphism and the Hamiltonian constraint implementing the space-time diffeomorphisms. Since one does not know how to carry out the Dirac (or otherwise) quantization of this constrained system, one looks for highly symmetric class of space-times, does the symmetry reduction classically and quantizes the left over finitely many degrees of freedom. In the context of so called Bianchi models, the gravitational phase space is at most six dimensional. This is a ‘minisuperspace’ quantization and has become synonymous with ‘standard quantum cosmology’. [4] Here are its main conclusions.

- (i) The Hamiltonian constraint (quadratic in momenta) can be interpreted as an evolution equation which is second order in a suitably chosen ‘time’ variable. This is intimately connected with the ‘problem of time’. In a 4-diffeomorphism invariant theory one has to interpret dynamics in terms of evolution relative to a ‘clock’ degree of freedom. For FRW models, the scale factor (a gravitational degree of freedom) is a natural choice of a ‘clock’. The evolution equation - Wheeler-De Witt equation - being second order in time has two independent solutions i.e. a non-unique wave function of the universe.
- (ii) When the scale factor vanishes, the inverse scale factor and hence the curvatures blow up, implying persistence of classical singularity.
- (iii) All geometrical quantities such as areas, volumes etc have continuous spectra.

Note that what is quantized is a *finite* dimensional phase space as nothing else could

be done. One was forced to do the symmetry reduction classically and then proceed to quantization. The Ashtekar reformulation in terms of connection variables offers a different alternative. One can quantize the infinite dimensional kinematical phase space and explore the possibility of doing symmetry reduction after quantization. Further more generically the spectra of areas , volumes etc turn out to be discrete indicating that this quantization is qualitatively different.

III. QUANTUM GEOMETRY FRAMEWORK

Let me briefly recall some of the basic steps.

(a) Choose a classical phase space: This involves choosing a three manifold, Σ , with or without boundary and/or asymptotic regions. The basic variables (for the gravitational sector) are $su(2)$ (Lie algebra of $SU(2)$ group) valued connection 1-forms, $A_a^i \tau_i$ and a densitised triad, E_i^a ('dual' to a 2-form). The classical configuration space is the space of all smooth (real analytic) connections satisfying appropriate boundary conditions, \mathcal{A} . The space modulo gauge transformations is denoted as \mathcal{A}/\mathcal{G} . The phase space is the cotangent bundle of the configuration space. On this phase space we have the usual Gauss law, the diffeomorphism and the Hamiltonian constraints.

(b) There is a natural class of functions on \mathcal{A}/\mathcal{G} namely, $Tr(P \exp \int_\gamma A)$. These are labeled by closed loops in Σ , are gauge invariant and diffeomorphism invariant. One now chooses to look for a Hilbert space on which these functions are represented by multiplicative operators. This is achieved by using the commutative C^* algebra of these functions and choosing one of the representations of this algebra. This gives $H_{kin} = L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_{AL})$. In other words, the Hilbert space is the space of square integrable complex functions on the 'quantum configuration space' \mathcal{A}/\mathcal{G} square integrable with respect to the Ashtekar-Lewandowski measure constructed from the Haar measure

on $SU(2)$. A convenient description of the Hilbert space is obtained in terms of the so called spin-network functions. [5]

In Σ consider closed graphs γ . Associate with each edge e_i a representation π_i of $SU(2)$ and associate with each vertex v_α an intertwiner (contractor/invariant tensor), C_α . For each $A \in \bar{\mathcal{A}}$ one has, by definition, $g_i = A(e_i)$, the generalized holonomy. Define,

$$\psi_\gamma(A) = "Tr" \Pi_{edges} \pi_i(g_i) \Pi_{vertices} C_\alpha \quad (1)$$

These are the spin network functions - functions on \mathcal{A}/\mathcal{G} - labeled by graphs, representations and contractors. The set of all such functions forms an orthonormal set which is dense in H_{kin} . In practice one defines various operators by their actions on these and extends them to H_{kin}^Σ . This dense space is also called the space of cylindrical function.

(c) In order to accommodate the possibility that the zero eigenvalue of constraints could be in a continuum (i.e. a generalized eigenvalue) one introduces a ‘rigging’ - $\Omega \subset H_{kin} \subset \Omega^*$. Ω is the dense space above while Ω^* is the space of continuous linear functional on Ω . The important point is that the physical states generically belong to Ω^* . One says that physical states are distributional. [6]

(d) For matter sector analogous constructions based on suitable C^* algebras are made. These are available for all the usual scalar, spinor, gauge fields. The kinematical Hilbert spaces constructed here differ from the usual Fock spaces. The crucial point of these constructions is to have no dependence on background space-time geometry. [7]

IV. QUANTUM SYMMETRY REDUCTION

One would now like to specialize this frame work to the cosmological context of highly symmetric space-times. Since the configuration space variable are now connections, the

notion of symmetry requires that the connection transformed under a symmetry diffeomorphism of Σ to be gauge equivalent to the original connection. The first task is to characterize such symmetric connections precisely. This has already been done and can be summarized as (simplifying a bit for brevity): [8,9]

If S is a symmetry group (compact Lie group) and F a (Lie) subgroup of S , then

- (a) $\Sigma \sim B \times S/F$, $B \sim \Sigma/S$ is the space of orbits of S -action on Σ ;
- (b) Symmetric connections on Σ are completely characterized by a (reduced) connection on B together with a set of (Higgs) scalars on B , possibly satisfying further constraints.

For example, for the Bianchi class A models, S is one of the Bianchi groups with structure constants satisfying $C_{JI}^I = 0$ while $F = \{e\}$. Σ is the group manifold of S while B is a single point of Σ . In terms of the Maurer-Cartan 1-forms ω^I and the corresponding left invariant vector fields X_I one has,

$$A_a^i = \Phi_I^i \omega_a^I \quad E_i^a = P_i^I X_I^a \quad \{\Phi_I^i, P_j^J\} = 8\pi G \gamma \delta_j^i \delta_I^J \quad (2)$$

Φ_I^i are the scalars, P_i^I are the conjugate momenta, γ is the Barbero-Immirzi parameter and the indices i ($su(2)$), I (Lie algebra of Bianchi group) both take three values. If in addition to spatial homogeneity one also has isotropy then the scalars satisfy further conditions whose solution is $\Phi_I^i = c \delta_I^i$ while the conjugate momenta satisfy $P_i^I = p \delta_i^I$. The constraint expressions can be likewise simplified and expressed in terms of the scalars and their conjugates. This is the classical symmetry reduction. If one proceeds with quantization in a traditional manner then this is very similar to the usual minisuperspace quantization apart from the phase space variables being different. The results are also similar. [10]

However an alternative quantization is possible. Instead of requiring the scalars to be well defined operators one can take their exponentials, the so called point holonomies, as well defined operators. Then, similar to the general framework of quantum geometry (polymer representation), one constructs a new H_{kin}^B . One can immediately ask why one should do this? Does this Hilbert have anything to do with the H_{kin}^Σ of the full theory? The answer turns out to be yes! Bojowald and Kastrup show that the (cylindrical) states in H_{kin}^B can be identified with those distributions in Ω_Σ^* whose support consists of precisely the classical (smooth) symmetric connections. [9] Recall that the physical states of the full theory are supposed to reside in Ω_Σ^* . A natural identification of symmetric (distributional) states of the full theory is via the properties of their support. The result shows that such quantum states can be alternatively be dealt with by working with the cylindrical states of H_{kin}^B . This provides a justification for the alternative (holonomy based) quantization of the symmetry reduced theory. It also makes available the tools of the general framework in a simplified context. This is what is referred to as ‘quantum symmetry reduction’. Note that this is quite general and not restricted to a cosmological context.

V. BIANCHI CLASS A MODELS

The strategy now is step by step adaptation of the general framework. While majority of steps are identical, there are also crucial new inputs needed particularly when additional symmetry such as isotropy is at work. I am including only the minimal details necessary to communicate the final results.

For classical configurations, point holonomies are just the group elements obtained by exponentiating the Lie algebra valued scalars, $u(\Phi_I) \equiv \exp\{\Phi_I^i \tau_i\} \in SU(2)$. Distributional scalars do just that, they associate with each vertex, an $SU(2)$ group element. For general anisotropic case the kinematical Hilbert space turns out to be

$H_{kin} = L^2(SU(2)^3, d\mu_{Haar}^3)$. [11] There are three copies since I takes three values. For gauge invariant functions, a convenient basis is provided by the usual spin-network functions constructed from graphs γ in the group manifold of S with a single vertex of order 6 and three (closed) edges corresponding to the three left invariant vector fields. Using these one defines the operators corresponding to the conjugate momenta and then proceeds to build the constraint operators. Following the Thiemann approach, the Hamiltonian constraint is expressed in terms of the volume operator together with various commutators of holonomies with the volume operator. So the main problem is to define a volume operator and obtain its spectrum. The full spectrum is not available in the general case. However, for the homogeneous and isotropic models (Bianchi I and IX), spectrum of volume operator has been determined.

When isotropic case is considered, the scalars have to satisfy further conditions. Naively one would expect that since we now have a single scalar, spin-network functions associated with graphs with a single closed edge (and a single vertex) should suffice. This turns out to be false. Although the H_{kin} in this case does turn out to be $L^2(SU(2), d\mu_{Haar})$, it contains more gauge invariant functions than what a graph with single edge could supply. Bojowald determines the extra functions needed thereby obtaining an explicit orthonormal basis for the kinematical Hilbert space. A volume operator is now defined explicitly and its spectrum obtained. [12] The eigenfunctions and eigenvalues are given by (in the gauge invariant sector),

$$\begin{aligned}\chi_j(c) &= \frac{\sin((j + \frac{1}{2})c)}{\sin(c/2)} \quad j = 0, 1/2, 1, \dots \\ \zeta_j(c) &= \frac{\cos((j + \frac{1}{2})c)}{\sin(c/2)} \quad j = 0, 1/2, 1, \dots \\ \zeta_{-\frac{1}{2}} &= \frac{1}{\sqrt{2}\sin(c/2)} \\ V_j &= (\gamma\ell_p^2)^{3/2} \sqrt{\frac{j(j + \frac{1}{2})(j + 1)}{27}}\end{aligned}\tag{3}$$

It turns out to be more convenient to choose a slightly different orthonormal basis

which we now denote as: [13]

$$|n\rangle \equiv \frac{e^{i\frac{n}{2}c}}{\sqrt{2}\sin(c/2)}, \quad n \in \mathbf{Z} \quad (4)$$

These are also eigenstates of the volume operator with eigenvalues $V_{\frac{|n|-1}{2}}$. Now the Hamiltonian constraint can be defined by its action on these states. In general it has the form:

$$\hat{H}|n\rangle = \sum_{k=-L}^L A_n^k |n-k\rangle. \quad (5)$$

The physical states, $|s\rangle = \sum_n s_n |n\rangle$, which belong to the kernel of \hat{H} , have conditions on the coefficients s_n of the form,

$$\sum_{k=-L}^L A_n^k s_{n+k} = 0. \quad (6)$$

This is a difference equation for the s_n coefficients. The order of the equation, $2L$, depends on the Bianchi type while the coefficients A_n^k depend on the details of the Hamiltonian constraint and the factor ordering chosen. [14,15,18] For the flat isotropic model, the order of the equation is 16. The (*non-symmetric*) ordering chosen is such that the coefficient of s_0 always vanishes. One can introduce matter sector and add its contribution to the Hamiltonian constraint. Denoting matter symbolically by φ the equations are modified by putting in a φ dependence in s_n and adding a term of the form $\hat{H}_\varphi(n)s_n(\varphi)$. At this stage, particular form of matter and its couplings are not detailed. It is sufficient to note that the matter part of the Hamiltonian operator is diagonal with respect to the (gravitational) states $|n\rangle$. This specifies the physical states of the loop quantum cosmology. [13,16]

VI. DYNAMICAL INTERPRETATION, ABSENCE OF SINGULARITY AND UNIQUENESS OF SOLUTION

In order to interpret the physical states, in particular to explore what ‘happens’ to the classical singularity, one needs to view the physical states obtained above as solutions

of a ‘time evolution’ equation. In a generally covariant theory of space-time there is no external, inert ‘time’ variable. One can at best single out one of the degrees of freedoms as a ‘clock’ and view functions of this (gravitational or matter) degree of freedom as functions of ‘time’. While there is no a priori given clock, in a given context there can be a natural choice. For instance, for the FRW cosmologies, the scale factor in the metric is one such choice. It is in terms of this choice that one says that there is a singularity at the vanishing value of the scale factor (curvatures blow up). In the present formulation, absolute value of the conjugate momentum, p , corresponds to the square of scale factor. The spectrum of $|\hat{p}|$ consists of eigenvalues $(j+1/2)$ with eigenfunctions $\chi_j(c), \zeta_j(c)$. The zero eigenvalue of $|\hat{p}|$ occurs for $j = -1/2$ which is a non-degenerate eigenvalue. (The volume operator consists of this operator together with another commuting operator with eigenvalues $j(j+1)$. This makes the zero eigenvalue of the volume operator three fold degenerate.) In terms of the $|n\rangle$ basis, $n = 0$ corresponds to vanishing scale factor. One therefore takes, the label $n \in \mathbf{Z}$ as a time label. The physical state condition can now be viewed as specifying a discrete time evolution. [15,13,16]

Restricting to the isotropic, flat models, the evolution equation is of order 16 (and consists of the $n \pm 8, n \pm 4$ and n terms only). The coefficients are such that $A_n^k = 0$ if and only if $n + k = 0$. Furthermore, $\hat{H}_\varphi(n=0) = 0$ as well. This has two consequences. The $s_0(\varphi)$ never appears in the equation. Therefore it is neither determined by nor determines any other $s_n(\varphi)$. The state $|0\rangle$ is thus orthogonal to all other physical states (evolving solutions). This is precisely the state which corresponds to the zero eigenvalue of the scale factor operator. Thus one sees that unlike the classical case where evolution through vanishing scale factor is not possible, the quantum evolution equation does not suffer such a breakdown. In this sense the classical singularity is absent in the evolving states. The second consequence is that one gets conditions on the initial data - the choice of 16 coefficients $s_{-16}, s_{-15}, \dots, s_{-1}$ (say). This is because the equation can not determine s_0 since its coefficient vanishes. Thus one can conclude that the classical

singularity is avoided by the evolving solutions and that there are 15 instead of 16 independent solutions. [13,16]

This seems worse than the standard quantum cosmology with only two independent solutions! Quite independently, one has also to understand the discrete time evolution in terms of more familiar continuous time evolution, at least when one hopes to be in classical regime. Clearly, the volume and the scale factor eigenvalues become large compared to the Planck scale values when $n \gg 1$. One expects a solution to be classically interpretable when for large n , $s_{n+m}(\varphi)$ can vary significantly from $s_n(\varphi)$ when m is also large compared to 1 but remains almost the same when m is comparable to 1. One can now ask: how many solutions exhibit this behavior? In a precise formulation of such a behavior, the Barbero-Immirzi parameter γ come very handy. Note that for large $n = 2j + 1$, $V_j \rightarrow (\gamma \ell_p^2 |n|/6)^{3/2} \sim (a^2)^{3/2}$. We can use this to define the scale factor as a function of n as: $a^2(n) \equiv (n\gamma)\ell_p^2/6$. Now change in $a^2(n)$ as n changes by 1 will be infinitesimal if $\gamma\ell_p^2 \rightarrow 0$. Thus we can mimic continuous evolution with respect to the scale factor by considering the formal limit $\gamma \rightarrow 0, n \rightarrow \infty$ keeping γn a constant. We are keeping ℓ_p fixed in this and so are in the quantum domain still. If a solution $\{s_n(\varphi)\}$ has a limiting value in the above limit then it is said to be ‘pre-classical’. [16] The question now becomes as to how many evolving solutions are pre-classical? For large n , the equations itself becomes an equation with constant coefficients and is easy to analyze. The result is that, of the sixteen solutions, only two are pre-classical. There is still the constraint on the initial conditions which reduces these two to a single solution. Thus while there are many evolving solutions (all avoiding the singularity), only one of these can mimic a classical evolution. [16]

From the black hole entropy computations, γ is fixed to be a constant of order one. The $\gamma \rightarrow 0$ limit noted above is to be thought of as a formal device to single out solutions mimicking classical evolution. The presence of the Barbero-Immirzi parameter however

offers a new limit to be explored apart from the classical limit with $\ell_p \rightarrow 0$. Bojowald has further shown that in the $\gamma \rightarrow 0$ limit, referred to as ‘continuum limit’ in the sense that the discrete structure of quantum geometry can be ignored, one recovers the standard quantum cosmology. [17] This shows also that it is the specific, discrete structure of quantum geometry that is responsible for avoiding the singularity and selecting a unique (pre-classical) solution and not just any quantization procedure.

VII. CONCLUDING REMARKS

This is a first explicit application of the highly abstract framework of quantum geometry to a context of physical interest, particularly addressing the issue of classically indicated singularity. Not only does it meet the expectation that in a quantum theory of gravity, classically indicated singularities should be absent, it provides a quantitative means to estimate how rapidly quantum geometry picture goes over to the classical picture. In the quantization procedure, many conceptual and technical assumptions have gone in (eg choice of polymer representation). In a sense the classical space-time picture has been ‘mutilated’ quite a bit. Therefore it could have happened that the none of final set of solutions could be interpreted in classical terms thereby recovering the classical theory. Not only this does *not* happen but one gets a unique solution displaying the classical picture. It also brings out an intriguing role played by the Barbero-Immirzi parameter.

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